The Demand for Assets: Evidence from the Markov Switching Normalized Quadratic Model*

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Abstract:

In this paper we provide and illustrate a solution to the inter-related problems of estimation of asset demand functions, instability of money demand relations, and monetary aggregation. We use an econometric framework that very flexibly allows for changes in the coefficients of the asset demand functions and then quantifying the implications that this asset demand instability has for monetary policy and monetary aggregation. Our approach allows the estimation of asset demand functions in a systems context, using a flexible functional form for the aggregator function, based on the dual approach to demand system generation. However, instead of assuming that consumer preferences are fixed as in the neoclassical demand theory, we assume Markov regime switching, thus allowing for complicated nonlinear dynamics and sudden changes in the parameters of the asset demand functions and the underlying aggregator function. We use the monthly time series data on monetary asset quantities and their user costs, recently produced for the United States and maintained within the Center of Financial Stability (CFS), and the Normalized Quadratic (NQ) flexible functional form with Markov regime switching, to generate inference in terms of a full set of elasticities, simultaneously achieving consistency with the data generating process and economic and econometric regularity. We find evidence that our five-regime NQ model provides a better fit of the actual data than a single regime model, that the asset demand specifications exhibit instability between regimes, but are rather stable within regimes, and that the assets are in general Morishima substitutes with the Morishima elasticities of substitution always being below unity.

*JEL classification:* C32, C51, E41, E42, E51.

*Keywords:* Flexible functional forms; Demand systems; Markov switching; Divisia monetary aggregation.
1 Introduction

In this paper we provide and illustrate a solution to the inter-related problems of estimation of asset demand functions, instability of money demand relations, and monetary aggregation. We use an econometric framework that very flexibly allows for changes in the coefficients of the asset demand functions and then quantifying the implications that this asset demand instability has for monetary policy and monetary aggregation. Our approach allows the estimation of asset demand functions in a systems context, using a flexible functional form for the aggregator function, based on the dual approach to demand system generation. However, instead of assuming that consumer preferences are fixed as in the neoclassical demand theory, we assume Markov regime switching, thus allowing for complicated nonlinear dynamics and sudden changes in the parameters of the asset demand functions and the underlying aggregator function. We use the monthly time series data on monetary asset quantities and their user costs, recently produced for the United States and maintained within the Center of Financial Stability (CFS), and the Normalized Quadratic (NQ) flexible functional form with Markov regime switching, to generate inference in terms of a full set of elasticities, simultaneously achieving consistency with the data generating process and economic and econometric regularity.

Our work builds on a large body of literature, which Barnett (1997) calls the ‘high road’ literature, that takes a microeconomic- and aggregation-theoretic approach to the demand for assets. It follows the innovative works by Diewert (1974) and Barnett (1980, 1982, 1983) and utilizes the flexible functional forms approach to investigating the inter-related problems of estimation of asset demand functions and monetary aggregation. See, for example, Ewis and Fisher (1984), Serletis and Robb (1986), Serletis (1988, 1991), Barnett et al. (1992), Fisher and Fleissig (1997), Serletis and Shahmoradi (2007), Jadidzadeh and Serletis (2019), and Serletis and Xu (2020, 2021), among others. However, these papers postulated constant parameters in the underlying utility function (and the resulting demand system), and as Barnett and Kanyama (2003) put it, “when using real data, the consistency of the estimated coefficients of the demand system can be compromised if one wrongly assumes the constancy of the parameters, while they are actually random or varying over time. In this case the constant-coefficient model will not only fail to capture the possible long-run dynamics in the data but also will produce a poor approximation to the underlying data-generating process.”

In this regard, consumer preferences over financial assets seem to change quite dramatically in response to shocks that hit the economy. For example, commodity price shocks, large-scale events (such as wars and financial crises), changes in government policy (such as the introduction of inflation targeting), and technological and institutional changes can induce significant shifts in consumer tastes and preferences over financial assets. For example, Guiso et al. (2017) find that qualitative and quantitative measures of risk aversion increase substantially after financial crises, and that fear is a potential mechanism that influences financial decisions, whether by increasing the curvature of the utility function or the salience
of bad outcomes. The change in behavior may be permanent, typically known as a ‘structural break,’ temporary, or shift to another style of behavior, usually called a ‘regime switch.’ Also, Gordon and St-Amour (2000) show that the regime-switching preferences of households could successfully resolve the equity premium, the risk-free rate, and the predictability puzzles. They also find that the regime-switching preferences of households are able to explain the sharp swings in assets prices that are characteristic of financial market data. Thus, what is required is a framework that is sufficiently flexible to allow for different types of behavior at different times and that also utilizes all of the available observations on the series to be used for estimating the demand system. Two classes of models that allow this to occur have been used so far in empirical demand analysis: the ‘time-varying coefficient model’ — see Barnett and Kanyama (2003) and Mazzocchi (2003) — and the ‘Markov switching model’ — see Allais and Nichèle (2007). The former assumes that consumer preferences change in every period (which might not be the case in the real world), whereas the latter assumes a switching mechanism from one state of preferences to another that is controlled by an unobserved variable governed by a Markov process.

In this paper we take the Markov switching approach, associated with Hamilton (1989), which has been widely followed in the analysis of economic and financial time series — see, for example, Sims and Zha (2006). We model the locally flexible NQ expenditure function, introduced by Diewert and Wales (1988), as a function of an unobserved regime-shift variable, governed by a first-order \( m \)-state Markov process. We use a statistical criterion to optimally select the number of preference regimes and generate inference in the context of a five-regime NQ demand system. In doing so, we pay explicit attention to the theoretical regularity conditions of positivity, monotonicity, and curvature, because as Barnett (2002, p. 199) put it in his Journal of Econometrics Fellow’s opinion article, without satisfaction of all three theoretical regularity conditions “... the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid.” We generate inference in terms of a full set of elasticities — income elasticities, own- and cross-price elasticities, and the Allen-Uzawa and Morishima elasticities of substitution. We find evidence that our five-regime NQ model provides a better fit of the actual data than a single regime model, and that the assets are in general Morishima substitutes, with the degree of substitution being regime-dependent and the Morishima elasticities of substitution always being below one.

We also find that the asset demand specifications are stable within regimes (locally stable), but exhibit some instability between regimes (in terms of swings in the elasticities of substitution, reflecting the changes in preference structure for asset demand) during periods of erratic monetary policy, financial innovation, and regulatory changes. In the terms of our framework, which is based on a strong link between neoclassical microeconomic theory and econometric implementation, if the elasticities of substitution between assets are small, the asset demand functions are stable and predictable, and the central bank’s ability to accurately predict how much of its wealth the public will want to hold in the form of money and
near-money assets is enhanced. In this case, velocity is easy to predict, and the central bank can target key monetary aggregates to accommodate the demand for money and near monies and affect general macroeconomic variations, as it is in the quantity theory of money. If, however, the elasticities of substitution between assets are large, and thus the asset demand functions are unstable, then velocity is unpredictable, and the supply of money will not be closely linked to aggregate spending. In this case, the interest rate might be a better measure of the stance of monetary policy than the money supply.

The rest of the paper is organized as follows. Section 2 briefly sketches related neoclassical demand theory and applied consumption analysis. Section 3 discusses Markov regime switching demand systems and analytically proves the invariance of the maximum likelihood estimator with respect to the choice of the good deleted and the ordering of the remaining equations in the estimated subsystem. Section 4 presents the NQ expenditure function, derives the associated system of budget share equations, and discusses the Diewert and Wales (1988) method of imposing global concavity with the objective of achieving theoretical regularity. Section 5 discusses the data and section 6 presents the empirical results. Section 7 discusses the stability of the asset demand functions, section addresses the robustness of our results, and section 9 discusses the implications for monetary aggregation. The final section contains concluding remarks.

2 Neoclassical Demand Theory

Let’s consider an economy with identical households whose direct utility function is weakly separable (a direct tree) as follows

\[ U = f(c, \ell, u(x)) \]  

where \( c \) is a \( \kappa \)-vector of consumption goods, \( \ell \) is leisure time, and \( x \) is a \( n \)-vector of the services of monetary assets (assumed to be proportional to the stocks). The utility-tree structure (1) is treated as a maintained hypothesis in this paper, as is the case with a large number of studies in the literature. It implies that the demand for monetary services is independent of relative prices outside the monetary group, since the aggregator function \( u(x) \) is weakly separable in \( x \) from \( c \) and \( \ell \) iff

\[ \partial \left[ (\partial u/\partial x_i) / (\partial u/\partial x_j) \right] / \partial x_k = 0, \quad x_k = c_1, \ldots, c_\kappa, \ell \]

where the expression in brackets is the marginal rate of substitution between monetary assets \( i \) and \( j \). In this regard, Hjertstrand el at. (2016) find that consumption goods, leisure and money are weakly separable, using an advanced revealed preference test.

Under the assumptions made, we can focus on the details of the demand for the services of monetary assets, ignoring the services of consumption goods, \( c \), and leisure, \( \ell \), in terms of
the following consumer problem

$$\max_x u(x) \text{ subject to } \pi x = y$$

where $y$ is the expenditure on the services of monetary assets and $\pi = (\pi_1, ..., \pi_n)'$ is the vector of monetary asset user costs (or rental prices), with the $j$th element given as in Barnett (1978) by

$$\pi_j = \frac{R - r_j}{1 + R}, \quad j = 1, ..., n$$

where $R$ is the yield on an alternative asset (called benchmark asset) and $r_j$ is the yield on the $j$th asset.

The solution of the first-order conditions for utility maximization is the demand system

$$x = x(\pi, y)$$

and the associated indirect utility function $h(\pi, y) = u(x(\pi, y))$. The demand system can be expressed in budget share form $s = (s_1, ..., s_n)'$, where $s_j = \pi_j x_j(\pi, y)/y$ is the expenditure share of asset $j$. Since Marshallian demands are homogenous of degree zero in $(\pi, y)$, implying the absence of money illusion, the demand system can also be written in budget share form in terms of expenditure-normalized prices (user costs) as follows

$$s = s(v)$$

where $v = (v_1, ..., v_n)'$ is the vector of expenditure-normalized user costs, with the $j$th element being $v_j = \pi_j/y$, $j = 1, ..., n$.

2.1 Velocity Function

As shown in Barnett and Serletis (2000, p. 96-97), our framework can also be used to derive the velocity function. In particular, in the first stage of the two-stage optimization, the consumer maximizes

$$U = f(c, \ell, M)$$

subject to the full income constraint

$$q'c + w\ell + \Pi M = Y$$

where $q$ is the vector of the prices of $c$, $w$ is the wage rate, $M = u(x)$ is the monetary aggregate over $x$, $\Pi$ is the corresponding user cost aggregate, and $Y$ is the total current period expenditure on goods, leisure, and monetary assets.

The solution function for $M$ will be of the form

$$M = \Psi(Y, \Pi, q, w).$$
with the expenditure on the services of monetary assets, the $y$ variable in equation (2), being

$$y = \Pi M = \Pi \Psi(Y, \Pi, q, w).$$

If $U$ is linearly homogeneous, then equation (4) can be written as

$$M = Y \Phi(\Pi, q, w)$$

so that $[\Phi(\Pi, q, w)]^{-1}$ is the velocity relative to the ‘income’ variable $Y$.

It is particularly interesting to observe that although our income variable is not gross domestic product (GDP), our internally consistent approach is also congruent with reality, as we control for economic growth by having asset demand functions that depend on the total expenditure on goods, leisure, and monetary assets, $Y$, as in equation (4). Of course, this is different from those demand for money studies that use real money balances as a fraction of GDP as the dependent variable.

### 3 Markov Switching Demand Systems

In order to estimate demand systems such as (3), a stochastic version must be specified. It is typically assumed that the observed share in the $j$th equation deviates from the true share by an additive disturbance term $\epsilon_j$. It is also assumed that $\epsilon_t \sim F(0, H, g(\epsilon_t' H^{-1} \epsilon_t))$, where $F$ is a distribution from the family of elliptical distributions, $0$ is the null vector, $H$ is the $n \times n$ symmetric positive definite error covariance matrix, and $g$ is the density generator of $\epsilon_t$. With the addition of additive errors, the share equation system (3) can be written as

$$s_t = s(\vartheta | v_t) + \epsilon_t \quad (5)$$
$$\epsilon_t \sim F(0, H, g(\epsilon_t' H^{-1} \epsilon_t)) \quad (6)$$

where $s(\vartheta | v) = (s_1(\vartheta | v), ..., s_n(\vartheta | v))'$, $s_i(\vartheta | v)$ and $\vartheta$ is the parameter vector to be estimated.

In this paper, we follow Hamilton (1989) and model a demand system as a function of an unobservable regime-shift variable, $z_t$, such that the demand system (5)-(6) takes the following form

$$s_t = s(\vartheta | v_t, z_t) + \epsilon_{t,z_t} \quad (7)$$
$$\epsilon_{t,z_t} \sim F(0, H_{z_t}, g_{z_t}(\epsilon_{t,z_t}' H_{z,t}^{-1} \epsilon_{t,z_t})) \quad (8)$$

where the $z_t$ variable describes the unobservable ‘regime’ (or ‘state’) that the demand for assets was in at time $t$. The main feature of a regime switching demand system is that the parameters $\vartheta$ and errors $\epsilon_{t,z_t}$ are regime dependent. In fact, equation (8) allows the
demand for assets to have different unconditional volatilities in different regimes and the regime-dependent errors to have different distributions in different regimes. For example, $g$ could be the generator of a normal distribution in one regime and of a Laplace distribution in another regime.

Changes between regimes are assumed to be the result of a first-order, homogeneous, $m$-state Markov process governed by the transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{pmatrix}$$

where $p_{ji}$ the probability that consumer preferences are in regime $j$ in period $t$ given that they were in regime $i$ in period $t-1$, $p_{ji} = p(z_t = j \mid z_{t-1} = i)$, $i, j = 1, \ldots, m$. Also, since at any given time consumer preferences must be in one of the $m$ regimes, it must be that $\sum_{j=1}^{m} p_{ji} = 1$, for all $i$.

Since the regime-dependent errors satisfy the adding up condition, $\sum_{i=1}^{m} \epsilon_{it,zt} = 0$, we have

$$E(\epsilon_{t,zt} \epsilon_{t,zt}') = H_{zt} i = 0$$

where $i$ is a $n \times 1$ vector of ones. Thus, the covariance matrix of the Markov switching demand system is singular no matter which regime it belongs to.

### 3.1 Invariance

The fact that $s_{t,zt}$ (and therefore $\epsilon_{t,zt}$) satisfy the adding up condition (i.e., the budget shares sum to 1), imply that the disturbance covariance matrix, $H_{zt}$, is singular. This introduces a technical problem when the demand system is estimated, since either generalized least squares or maximum likelihood (ML) needs to invert the covariance matrix, $H_{zt}$. In the case of single-regime demand systems, Barten (1969) and McLaren (1990) show that maximum likelihood estimates can be obtained by arbitrarily dropping any equation in the system, assuming a normal distribution for the errors. McLaren (1990) also establishes invariance by virtue of observational equivalence of the subsystems with different deleted equations. In what follows, we generalize these results to the case of Markov switching demand systems.

**Theorem 1** The subsystems of (7) and (8) obtained by deleting a different equation, are observationally equivalent under the maximum likelihood estimator.

**Proof of Theorem 1** See Appendix 1.
Next we show that the invariance result also holds irrespective of the ordering of the equations in each of the subsystems obtained by deleting an arbitrary equation.

**Theorem 2** The subsystems of (7) and (8) obtained by deleting the same equation, but have different ordering of the remaining equations, are observationally equivalent under the maximum likelihood estimator.

**Proof of Theorem 2** See Appendix 2.

**Corollary 1** The subsystems of (7) and (8) obtained by deleting a different equation, are observationally equivalent under the maximum likelihood estimator, and the equivalence holds, irrespective of the ordering of the remaining equations in each of the subsystems.

According to Corollary 1, the invariance of the maximum likelihood estimator holds, irrespective of which equation is deleted, and also irrespective of the ordering of the remaining equations in each of the subsystems.

### 4 The NQ Expenditure Function

Our objective is to estimate demand systems derived from the indirect utility function \( h(p, y) \). We prefer the indirect utility function approach, because prices enter as exogenous variables in the estimation process and the demand system is easily derived by applying Roy’s identity. In this paper, we use the normalized quadratic function, introduced by Diewert and Wales (1988). This is a flexible functional form able to approximate an arbitrary, twice-continuously-differentiable preference function to the second order.

The normalized quadratic expenditure function is defined as follows

\[
C(p, u) = \theta'p + \left( b'p + \frac{1}{2} \frac{p'Bp}{\alpha'p} \right) u
\]

where \( \theta = (\theta_1, \ldots, \theta_n)' \), \( b = (b_1, \ldots, b_n)' \), and the elements of the \( n \times n \) matrix \( B \equiv [\beta_{ij}] \) are the unknown parameters to be estimated. To ensure the flexibility and Gorman polar form of the NQ form, we follow Diewert and Wales (1988) and impose the following restrictions

\[
\sum_{i=1}^{n} \alpha_ip_i^* = 1, \quad \alpha_i \geq 0 \quad \forall i
\]

\[
\sum_{i=1}^{n} \theta_ip_i^* = 0
\]
and
\[ \sum_{j=1}^{n} \beta_{ij}p_j^* = 0 \quad \forall i \quad \text{and} \quad \beta_{ij} = \beta_{ji}, \quad \forall i, j \quad (12) \]

where \( p^* \gg 0_n \) is a reference (or base-period) vector of normalized prices, determined in such a way that \( p^* = 1_n \). The non-negative vector \( \alpha = (\alpha_1, \ldots, \alpha_n)' \) can be predetermined as a vector of ones \( (\alpha = 1_n) \) with no loss in generality for restriction (10) — see Diewert and Wales (1988) for more details.

The NQ demand system in budget share form is
\[ s_i(v) = \theta_i v_i + \frac{b_i + \left( \sum_{j=1}^{n} \beta_{ij}v_i \right)}{\left( \sum_{i=1}^{n} \alpha_i v_i \right)} - \frac{1}{2} \left( \frac{\alpha_i \sum_{k=1}^{n} \sum_{j=1}^{n} \beta_{kj}v_k v_j}{\left( \sum_{i=1}^{n} \alpha_i v_i \right)^2} \right) \times \left( 1 - \sum_{i=1}^{n} \theta_i v_i \right) v_i \quad (13) \]

with \( 2n + n(n + 1)/2 \) parameters and \( 2n - 2 + n(n - 1)/2 \) linearly independent parameters to be estimated.

### 4.1 Theoretical Regularity

The concavity of the NQ expenditure function may not be satisfied, in the sense that the estimated \( B \) matrix may not be negative semidefinite. In that case, we follow Diewert and Wales (1988), and impose global concavity by setting \( B = -KK' \), where \( K = [k_{ij}] \) is a lower triangular matrix. For example, in the case with three goods \( (n = 3) \), concavity of the NQ expenditure function can be imposed by replacing the elements of \( B \) in (13) by the elements of \( K \), as follows

\[ \begin{align*}
\beta_{11} &= -k_{11}^2 \\
\beta_{12} &= -k_{11}k_{12} \\
\beta_{22} &= -(k_{12}^2 + k_{22}^2).
\end{align*} \]

The other elements of \( B \) can be recovered using restriction (12) as follows

\[ \begin{align*}
\beta_{13} &= -(\beta_{11} + \beta_{12}) \\
\beta_{23} &= -(\beta_{12} + \beta_{22}) \\
\beta_{33} &= \beta_{11} + 2\beta_{12} + \beta_{22}.
\end{align*} \]
5 Data and Aggregation Issues

We use the monthly time series data on U.S. monetary asset quantities and their user costs maintained within the Center of Financial Stability (CFS) program Advances in Monetary and Financial Measurement (AMFM). See http://www.centerforfinancialstability.org and Barnett et al. (2013) for a detailed discussion of the data and the methodology for the calculation of user costs. The sample period is from 1974:6 to 2019:11 (a total of 546 observations). As we require real per capita quantities for the empirical work, we divide each quantity series by the CPI (all items) and total population.

It is to be noted that our dependent variable is real money balances per capita, and not real money balances as a fraction of GDP (or its inverse, i.e., money velocity) as in the standard literature on money demand which consistently takes GDP as the relevant normalizing variable. Our approach is based on neoclassical microeconomic aggregation and index number theory and, as Barnett and Serletis (2000, p. 582) put it, “insists upon internal coherence among data, theory, and econometrics.”

In particular, we model the demand for the 15 monetary assets listed in Table 1. In terms of our notation, $x = (x_1, ..., x_{15})$ is the vector of the 15 monetary asset quantities described in Table 1, $\pi = (\pi_1, ..., \pi_{15})$ is the corresponding vector of user costs, and $y = \pi'x$ is the total expenditure on the services of monetary assets. However, the estimation of a disaggregated demand system encompassing all 15 monetary assets is not possible with Markov regime switching, since the number of parameters to be estimated would be very large. For example, the NQ demand system with Markov regime switching has 270 free parameters in the case of two regimes and 408 in the case of three regimes. As Pudney (1980, p. 875) puts it, “lack of data and lack of sufficient independent variation within the sets of data that are currently available combine with the ‘curse of dimensionality’ that inevitably afflicts very detailed demand studies to make ‘unrestricted’ estimation of these demand responses a practical impossibility.”

To reduce the number of parameters, we separate the 15 monetary assets into four groups based on how the Fed allocates the assets into monetary aggregates, and write the consumer problem (2) as

$$\max_q u(q) \quad \text{subject to} \quad p'q = y \tag{14}$$

where $q = (q_1, q_2, q_3, q_4)$ is a vector of monetary subaggregates with

$$q_1 = f_1(x_1, x_2, x_3, x_4, x_5)$$
$$q_2 = f_2(x_6, x_7, x_8, x_9, x_{10})$$
$$q_3 = f_3(x_{11}, x_{12}, x_{13})$$
$$q_4 = f_4(x_{14}, x_{15})$$

and $p = (p_1, p_2, p_3, p_4)'$ is the corresponding vector of price indices. In (14), the aggregator functions $f_r \ (r = 1, 2, 3, 4)$ are interpreted as subaggregate measures of monetary services,
and are constructed as Divisia quantity indices corresponding to which there exist Divisia price indices, \( p = (p_1, p_2, p_3, p_4)' \).

We then approximate the expenditure function corresponding to (14) using the normalized quadratic form presented in (9), and estimate the NQ demand system given in (13). Moreover, we allow the parameters in (13) to change subject to an unobservable regime-shift variable \( z_t \), thus estimating a Markov switching NQ model. According to our Corollary 1, maximum likelihood estimates can be obtained by arbitrarily dropping any equation in the nonlinear model, to handle the implications of the assumption of a singular variance-covariance matrix.

6 Estimation of the Markov Switching NQ Model

In the general case with \( n \) goods and \( m \) regimes, the NQ demand system with Markov regime switching has \( (2n - 2 + n(n - 1)/2)m \) free parameters (parameters estimated directly). In our case \( n = 4 \), and determining the state dimension is critical for estimation purposes.

6.1 Determination of the State Dimension

With the Markov switching NQ model, the statistical significance of an additional regime cannot be tested using standardized likelihood ratio tests. The reason is that the parameters associated with an additional regime are not identified under the null of \( m \) regimes, so that the likelihood function is flat with respect to the unidentified parameters. Hansen (1992) proposes a likelihood ratio test, but the critical value for the test statistic has to be empirically calculated using bootstrap methods. Implementing such a test is computationally intensive, unless the model is very simple. Given the computational difficulties in the large parameter space of the nonlinear NQ demand system, the Hansen (1992) test is econometrically intractable in our case.

In this paper, we use the Bayesian Information Criterion (BIC) to select the number of regimes. The BIC is an alternative measure that trades off the fit of the model against the number of estimated parameters for the regime switching model — see, for example, Hamilton (2016). In this regard, Psaradakis and Spagnolo (2003, 2006) show that such an approach works well for regime switching model selection, and the literature has already adopted this approach; see, for example, Herwartz and Lütkepohl (2014). To implement this method of determining the state dimension, we always estimate the Markov switching NQ model with the concavity restrictions imposed, when theoretical regularity (positivity, monotonicity, and curvature) is not satisfied by luck. In Table 2, we report the BIC values of the Markov switching NQ model under different regimes. It is to be noted that the single regime NQ model satisfies the theoretical regularity conditions at all sample points and thus the concavity conditions are not imposed. It is to be noted that the BIC values decrease as
we allow for more regimes, with the changes getting progressively smaller. In the end, we find that allowing one more regime in a five-regime model does not improve the performance of the model.

Overall, we find that the five-regime model \((m = 5)\) provides a better fit of the data. We also find that the five-regime model captures swings in the economy that the more parsimonious models cannot capture. In particular, we find that the three-regime model allows one more regime compared with the two-regime model, but the first two regimes of the three-regime model are not the same as those of the two-regime model. Also, the four-regime model suggests different regimes with more frequent switches, compared with the five-regime model. Overall, the more parsimonious two-, three-, and four-regime models do not capture the periods in which operational procedures by the Fed were changed, such as between 1979 and 1982.

In what follows we present evidence based on the Markov switching NQ model with five regimes \((m = 5)\). Figure 1 shows the smoothed probabilities of the five regimes, with these probabilities being reported at time \(t\), conditional on the full sample information. As can be seen, each regime lasts for several years, meaning that the structure of preferences does not change very frequently.

### 6.2 Maximum Likelihood Parameter Estimates

The parameter estimates of the five-state Markov switching NQ model, obtained using maximum likelihood (ML) estimation and with the curvature conditions imposed, are presented in Table 3, together with log likelihood values. For comparison purposes, in the first column of Table 3, we present the parameters of the single regime NQ model and in the remaining five columns those of the five-state NQ model. For each model, the invariance of the maximum likelihood estimator holds, irrespective of which equation is deleted, and irrespective of the ordering of the remaining equations in the subsystem (see Theorems 1 and 2 and Corollary 1). As can be seen in Table 3, most of the coefficients are statistically significant (based on individual tests).

### 6.3 Econometric Regularity

We also address econometric regularity issues by testing for a unit root in the residuals of each of the four share equations, using the Augmented Dickey-Fuller (ADF) test [see Dickey and Fuller (1981)] and the Dickey-Fuller GLS test [see Elliot et al. (1996)]. In doing so, we use the BIC to select the optimal lag length in the ADF test and we also use this selected lag length in the Dickey-Fuller GLS test. We conduct the unit root tests with no intercept and no trend, although our results are robust to the inclusion of deterministic components and alternative lag structures. The null hypothesis of a unit root can be rejected at the 1% level by both the ADF and DF-GLS test statistics for each of the four share equation...
residuals. We conclude that we achieve stationary residuals in modeling nonstationary asset shares and income normalized prices, and that econometric regularity is also satisfied.

6.4 Elasticities

To conduct empirical demand analysis, it is instructive to use different elasticities (income, own-, cross-, and substitution) for the Markov switching NQ model. The formula for the (uncompensated) own- and cross-price elasticities, $\eta_{ij}$, is

$$\eta_{ij} = \frac{\partial q_i}{\partial p_j} = \frac{p_j}{q_i}. \tag{15}$$

Using the homogeneity of degree zero in $(p, y)$ property of the Marshallian demand functions, the (uncompensated) income elasticities can be derived as

$$\eta_{iy} = -\sum_{j=1}^{n} \eta_{ij}. \tag{16}$$

The (compensated) Allen-Uzawa elasticities of substitution, $\sigma_{ij}^a$, can be calculated as

$$\sigma_{ij}^a = \frac{\eta_{ij} - \eta_{iy}}{s_j}. \tag{17}$$

However, as noted by Blackorby and Russell (1989), with more than two goods the Allen-Uzawa elasticity of substitution may be uninformative, and the (compensated) Morishima elasticity of substitution, $\sigma_{ij}^m$, is the correct measure of substitution

$$\sigma_{ij}^m = s_i \left( \sigma_{ji}^a - \sigma_{ii}^a \right). \tag{18}$$

It is to be noted that the Morishima elasticity of substitution $\sigma_{ij}^m$ looks at the impact on the optimal ratio $q_i/q_j$ when the price of asset $i$, $p_i$, changes holding the price of asset $j$, $p_j$, fixed. In particular, it measures the percentage change in $q_i/q_j$ induced by a percentage change in the relative price ratio, $p_i/p_j$; note that holding $p_j$ fixed, a percentage change in $p_i$ (equals a percentage change in the relative price $p_i/p_j$ and) will affect both $q_i$ and $q_j$. Goods will be Morishima substitutes ($\sigma_{ij}^m > 0$) if an increase in $p_i$ causes $q_i/q_j$ to increase and Morishima complements ($\sigma_{ij}^m < 0$) if an increase in $p_i$ causes $q_i/q_j$ to decrease.

We present the own- and cross-price elasticities and the Allen-Uzawa elasticities of substitution in an on-line Appendix. Here we focus on the asymmetrical Morishima elasticities of substitution, presented in Table 4 (along with $p$-values), evaluated at the mean of the data, for each of the four monetary assets, $q_1$, $q_2$, $q_3$, and $q_4$, and (for comparison purposes) the two models, the single regime NQ (in panel A) and the Markov switching NQ (in panel B). It is
to be noted that in general the differences between the Morishima elasticities of substitution based on the five-regime NQ model and those based on the single-regime NQ model are negative, suggesting that the single-regime NQ model overestimates the Morishima elasticities of substitution. Moreover, a $t$-test of the null hypothesis that the mean of the differences is zero, rejects the null hypothesis.

We also plot the complete set of the Morishima elasticities based on the Markov switching NQ in Figures 2-5. The red lines indicate the elasticities based on the Markov switching NQ model and the black lines the corresponding elasticities based on the single regime NQ model. Shaded areas indicate the five regimes. The evidence is that almost all Morishima elasticities of substitution are positive and statistically significant, and that there is on average low substitution among the monetary assets. We also note that although there are swings in these elasticities between regimes, reflecting the changes in preference structure for monetary asset demand, the monetary assets are Morishima substitutes ($\sigma_{ij}^m > 0$) in most regimes, with the Morishima elasticity numbers always being less than 1 and rather stable within regimes.

7 The Stability of Asset Demand Functions

The stability of asset demand functions is crucial to whether the central bank should target the money supply or interest rates. As Goldfeld and Sichel (1990, p. 300) put it, “the evidence that emerged, at least prior to the mid-1970s, suggested that a few variables (essentially income and interest rates, with appropriate allowance for lags) were capable of providing a plausible and stable explanation of money demand.” However, after 1973, the estimated money demand functions exhibited substantial instability, thus raising questions about the usefulness of the money demand function in the conduct of monetary policy. In trying to explain what happened, economists pointed to the rapid pace of financial innovation and to regulatory changes, in addition to revisiting empirical issues concerned with the specification of the money demand function and the choice of the dependent and independent variables. A vast literature devoted to these issues — see, for example, Judd and Scadding (1992), and more recently Lucas and Nicolini (2015), Benati et al. (2017), and Dai and Serletis (2019) — reveals that the instability of the traditional (simple-sum) money demand function has not yet been successfully explained.

In this paper, we follow the lead of Barnett (1980) and use aggregation theoretic money measures, and in particular Divisia monetary indices, which are fundamentally different from the Federal Reserve’s standard (simple-sum) measures. In doing so, we are aggregating the 15 monetary assets listed in Table 1 into 4 subaggregates in line with the Federal Reserve’s standard classification, thus internalizing the substitution within the monetary subaggregates. As already noted in the Introduction, in terms of our framework, if the elasticities of substitution between assets are small, then the asset demand functions and velocity are
stable and predictable, and the central bank can target key monetary aggregates in order to affect general macroeconomic variations, as it is in the quantity theory of money. If, however, the elasticities of substitution between assets are large, then velocity is unpredictable, and the monetary aggregates might not be a good measure of the stance of monetary policy. We would like to note that our model is still an approximation, in terms of capturing the data generating process, and that our estimates are also based on the ad hoc assumption about subaggregate measures of monetary services, although this assumption is consistent with how the Federal Reserve allocates assets into monetary aggregates.

In what follows, we discuss the five regimes detected by our five-state Markov switching NQ model, the stability of the asset demand functions, and the implications for the conduct of monetary policy.

7.1 Regime 1

**June 1974 to September 1979.** In the 1970s, the Fed was using the federal funds rate as its operating instrument and monetary aggregates as intermediate targets. However, during this period, monetary policy was procyclical (as it was in the 1950s and 1960s); it led to a positive association between money growth and the business cycle. The reason is that the Fed was actually more concerned with achieving interest-rate stability than targeting the monetary aggregates. When national income increased (decreased) and led to an increase (decrease) in market interest rates, the Fed intervened to buy (sell) bonds and lower (raise) interest rates to the desired levels. The resulting changes in base money would lead to procyclical changes in the money supply (increases in the money supply when the economy was expanding and decreases when the economy was in recession). According to our estimates of the Morishima elasticities of substitution in Figures 2-5, the degree of substitution between monetary assets was relatively low and asset demand functions were stable in this period.

7.2 Regime 2

**November 1979 to January 1983 and May 2014 to November 2019.** From October 1979 to October 1982, the Federal Reserve de-emphasized the federal funds rate as an operating instrument and conducted a policy of monetary targeting, using nonborrowed reserves (the monetary base minus discount loans) as the primary operating instrument and monetary aggregates as intermediate targets. In October 1982, however, the Fed abandoned monetary aggregates as a guide for monetary policy, and returned to a policy of smoothing interest rates. During the October 1979 to October 1982 period, according to our estimates of the Morishima elasticities of substitution in Figures 2-5, the degree of substitution between monetary assets was low and asset demand functions were stable. That is, according to our evidence, Paul Volcker (then chair of the Board of Governors) abandoned monetary aggregates at a time when asset demand functions were stable, providing reliable information
about the future course of the economy. This is consistent with the widely held view that Paul Volcker was not really serious about controlling any monetary aggregates during the October 1979-October 1982 period.

The period from May 2014 to November 2019 falls within our second regime. In this period, there is increased uncertainty and concerns that the Fed is now exposed to interest risk and credit risk, and that it has no exit strategy. As Wyplosz (2013, pp. 1) puts it, “a verdict on what they have achieved, and how, remains to be fully worked out, but at least they tried. When the economic and financial situation is deemed to have stabilized, they will have to reverse gears. The forthcoming exit strategy will be another historical first and as such, it requires a careful analysis of the daunting challenges that lie ahead.” There is also a general consensus now that, because the zero lower bound constraint on the policy rate is binding more frequently than it used to (for example, in 2003-2004 and in the aftermath of the global financial crisis), and lasts longer (over ten years, after October 2008), unconventional monetary policy tools will likely be kept in the Fed’s toolkits.

7.3 Regime 3

February 1983 to December 1989 and October 2011 to September 2013. The period from February 1983 to December 1989 is a period of significant financial innovation and regulatory changes in the U.S. banking sector, which can explain the instability of the asset demand functions in regime 3. As discussed in Lucas and Nicolini (2015), for many years after 1933, Regulation Q prohibited commercial banks from paying interest on bank demand deposits. In addition, Regulation Q allowed the Fed to impose interest rate ceilings on various types of deposits, including time deposits. Regulation Q was relaxed slightly in 1980, when banks were allowed to issue NOW accounts; these are checking accounts with some restrictions, such as a minimum average balance, that pay limited interest. Even to this day, U.S. banks are not allowed to pay interest on corporate checking accounts, but in 1982 they were allowed to issue interest-paying money market deposit accounts (MMDA), which could also be held by some corporations. NOW accounts were included in the M1 definition of the money supply, and the less liquid (because of restrictions on the number of transactions allowed per month) MMDA were included in M2 (together with savings deposits).

These regulatory changes significantly affected the demand for assets and the relative desirability of the different assets. As can be seen in Figures 2-5, the availability of close substitutes for money-like assets increased all the Morishima elasticities of substitution in regime 3. The higher degree of substitution decreased the sensitivity of M1 to changes in the interest rate, thus reducing the effectiveness of monetary policy in targeting monetary aggregates. During this period (in fact from October 1982 to mid-1993), the Fed used borrowed reserves (discount loan borrowings) as its main operating instrument and placed less emphasis on monetary aggregate targets. In February 1987, Alan Greenspan (then chair of the Board of Governors) announced that the Fed will no longer target M1, and that it
would switch its focus to M2, which it thought had a more stable relationship with economic activity. However, that relationship between M2 and economic activity also broke down in the early 1990s, and in July 1993 the Fed announced that it had completely abandoned monetary aggregates as a guide for monetary policy.

In terms of asset demand instability, the (short) period from October 2011 to September 2013 is similar to the February 1983 to December 1989 period; in fact, together they constitute regime 3. During the October 2011 to September 2013 period, the Fed was in a liquidity trap, after it pushed the policy rate to zero in October 2008 and was unable to lower it further. Without its primary policy tool, the Fed (and many other central banks around the world) started using non-interest-rate tools, known as nonconventional monetary policy: forward guidance, quantitative easing, and credit easing. These policies have been effective, but they also increased bank reserves to over $2.6 trillion from less than $50 billion before the financial crisis. This, together with the fact that the federal funds rate cannot be viewed as a measure of the stance of monetary policy when the zero lower bound constraint binds, creates uncertainty about future money supply movements. There has also been a monetary policy regime change, with the Fed, for the first time in its 100-year history, adopting an explicit 2% inflation target (based on ‘headline CPI’) in January 2012.

### 7.4 Regime 4

**July 1990 to March 1997.** Since the early 1990s, when the Fed abandoned the use of monetary aggregates as a guide for conducting monetary policy (in July 1993), it has been using the federal funds rate as the primary operating instrument. Regarding regime 4, the process of financial innovation that began in the early 1980s continued into the 1990s, but at a slower pace. Another important financial innovation during this period is the sweep technology. It was introduced in 1994, and enabled commercial banks to avoid the ‘tax’ from reserve requirements. In particular, at the end of a banking day, banks could sweep out of a corporation’s checking account any balances above a certain amount and invest in overnight securities that pay interest. The sweep accounts became very popular by the end of the 1990s, because the swept out funds were not subject to reserve requirements since they were not classified as checkable deposits. However, as can be seen in Figures 2-5, our asset demand functions are very stable in this period, as they were in the November 1979-April 1985 period.

### 7.5 Regime 5

**July 1997 to May 2011.** This is the period of the global financial crisis, the Great Recession, and of unconventional monetary policies. There was a false deflation scare in late 2002 and the Fed pushed the federal funds rate to 1% by July 2003 and keeping it there for a year. This low fed funds rate set off a number of bubbles. For example, house prices
(measured by the Case-Shiller home price index) increased by about 45% from 2003Q3 to their peak in 2006Q1, stock prices increased by about 66% from 2003Q1 until their peak in 2007Q1, and commodity prices increased by over 90% from 2003Q1 until their peak in 2008Q2. Also, during this period there were significant increases in leverage on Wall Street, with the then major Wall Street investments banks (Goldman Sachs, Morgan Stanley, Merrill Lynch, and Lehman Brothers) together averaging leverage levels of 30 to 1 before the financial crisis, up from 20 to 1 in 2003. From June 2004 to July 2006, the Fed raised the policy rate in 17 consecutive meetings, from 1% to 5.25%, perhaps in an attempt to lean against the price bubble in the housing market, but long-term interest rates in the U.S. (and around the world) declined for most of this period. Then, in the aftermath of the global financial crisis, the federal funds rate was quickly reduced to the zero lower bound and could not be driven below zero. The Fed (and many other central banks around the world) resorted to unconventional monetary policy in order to lower long-term interest rates and stimulate their economies.

Overall, our results suggest that although the Morishima elasticities of substitution are unstable between regimes, they are rather stable within regimes. That is, as there are shocks or changes (such as, for example, preference shifts), due to financial innovation, regulatory changes, and erratic monetary policy, economic agents update their beliefs and adjust their behavior, but monetary asset demand functions remain rather stable within regimes.

8 Robustness

We have accounted for instabilities in the demand for monetary assets using the Markov-switching NQ demand system. As already noted, such instabilities have been documented by a vast literature following Goldfeld’s (1976) study on the ‘case of the missing money.’ In this regard, Goldfeld and Sichel (1990), discuss several possible avenues which could be pursued in order to recover the stable demand for money which disappeared around the mid-1970s. Among the possibilities, they mention monetary aggregation approaches different from the standard simple-sum approach used by the Federal Reserve. Also, within the simple-sum framework of constructing monetary aggregates, they mention the possibility of expanding the standard definition of M1 (currency, demand deposits, and other checkable deposits), by adding other components performing economic functions which are sufficiently similar to those performed by either of the three ‘standard’ components. In particular, Goldfeld and Sichel (1990) list money market deposits accounts (MMDAs), money market mutual funds, and repurchase agreements.

Following Goldfeld and Sichel’s (1990) suggestions, Lucas and Nicolini (2015) expand the standard (simple sum) M1 aggregate with MMDAs — they call it NewM1. Although they do not perform econometric tests, their visual evidence appears to suggest stability in the relationship between NewM1 (as a fraction of nominal GDP) and a short rate. Also, Benati
et al. (2017) perform a battery of cointegration tests based on the standard M1 aggregate and the Lucas and Nicolini NewM1 aggregate, detecting no evidence of cointegration for the former and strong evidence of cointegration for the latter. They argue that the problem of instability of the U.S. demand for M1, documented most forcefully by Friedman and Kuttner (1992), originates from the Federal Reserve’s classification of MMDAs as part of M2, rather than as part of M1. They also argue that for several other countries in which MMDAs do not exist (such as, for example, Canada), the standard M1 aggregate works well, in the sense that cointegration tests detect a long-run equilibrium relationship between M1 velocity and a short rate. However, more recent work devoted to this issue — see, for example, Dai and Serletis (2019) — still detects evidence of instability of the NewM1 money demand function.

To investigate whether the instability in the demand for monetary assets originates from the fact that our Divisia aggregate $q_1$ (which is the Divisia equivalent of the standard simple-sum aggregate M1) does not contain MMDAs (which are instead contained in $q_2$, the Divisia equivalent of the simple-sum aggregate M2), we reconstruct our Divisia $q_1$ and $q_2$ aggregates by subtracting MMDAs from $q_2$ and augmenting the $q_1$ aggregate with MMDAs. To construct the new $q_1$ (Divisia NewM1) and $q_2$ aggregates, we need monthly data for MMDAs. The MMDAs series is available from December 1982 to August 1991, but the MMDAs and saving deposits series have been merged and are reported as one series since 1991 — see Barnett et al. (2013). To obtain a monthly MMDAs series after August 1991, we multiply the savings deposits (including MMDAs at commercial banks and thrift institutions) series after August 1991 by the average share of MMDAs in savings deposits (at commercial banks and thrift institutions) over the period from December 1982 to August 1991. Thus, our Divisia $q_1$ aggregate is now the equivalent of the Lucas and Nicolini simple-sum NewM1 aggregate — we call it Divisia NewM1. The question is whether the results with the Divisia NewM1 aggregate are different from those reported in the previous section. In fact, according to Lucas and Nicolini (2015) and Benati et al. (2017), we should expect the Markov-switching NQ demand system not to capture any evidence of instability in the demand for monetary assets.

Following the procedures discussed in the previous section, we estimate the Markov-switching NQ demand system with the new $q_1$ (Divisia NewM1), new $q_2$, $q_3$, and $q_4$ subaggregates, and present the empirical results in Table 5 and Figures 6-10. According to Table 5, the four-regime model is the preferred model. The smoothed probabilities, reported in Figure 6, provide clear evidence of regime switches over the sample period, related to the dynamics of financial markets. In particular, the first switch, from regime 1 to regime 2, is consistent with the time period when MMDAs were introduced into the financial market in 1983. The second switch, from regime 2 to regime 3, occurs at the beginning of 1995. An important financial innovation during this period is the sweep technology. It was introduced in 1994 and enabled commercial banks to avoid the tax from reserve requirements. In particular, at the end of a banking day, banks could sweep out of a corporation’s checking account any balances above a certain amount and invest in overnight securities that pay interest.
The sweep accounts became very popular by the end of the 1990s because the swept out funds were not subject to reserve requirements since they were not classified as checkable deposits. The most recent regime switch occurred after the 2008 financial crisis.

We also report the Morishima elasticities of substitution in Figures 7-10, in the same fashion as those in Figures 2-5 with the Divisia M1 aggregate. The results are in general consistent with the ones presented in the previous section. In particular, we find that the elasticities of substitution are positive most of the time, locally stable (stable within regimes), but exhibit instability between regimes, reflecting the changes in preference structure for monetary asset demand.

We conclude that unlike Lucas and Nicolini (2017) and especially Benati et al. (2017) who detect stability in the long-run demand for (simple-sum) M1 by simply putting MMDAs into M1, in our approach, adding MMDAs to the M1 aggregate does not restore stability of the demand for monetary assets. This is also consistent with the recent evidence by Dai and Serletis (2019) in their investigation of the welfare cost of inflation in the United States.

9 Implications for Monetary Aggregation

Currently the common practice among central banks is to construct monetary aggregates based on the simple-sum index, \( M = \sum_{j=1}^{n} x_j \), where \( x_j \) is one of the \( n \) liquid assets of the monetary aggregate \( M \). This simple-sum index views all liquid assets as dollar-for-dollar perfect substitutes. However, according to our estimates of the Morishima elasticities of substitution, although the degree of substitution between the liquid assets is regime-dependent, the assets are far from being perfect substitutes. It means that the simple-sum monetary aggregates used by central banks are inadequate, useless, and misleading. As Heckman and Serletis (2014, p. 1) recently put it, “the Federal Reserve Board and many other central banks around the world continue officially to produce and supply low quality monetary statistics, inconsistent with the relevant aggregation and index-number theory. This practice misleads central banks, as well as financial firms, mortgage lenders, and mortgage borrowers, regarding the levels of systemic risk in the economy, and also misleads economists regarding the appearance of instability of policy-relevant functions in the economy.”

There have been many attempts at properly weighting monetary assets within a monetary aggregate. However, it was Barnett (1980) who derived the theoretical link between monetary theory and aggregation and index number theory, by constructing monetary aggregates based upon Diewert’s (1976) class of superlative quantity index numbers. These aggregates are Divisia quantity indices — see Barnett et al. (1992), Barnett and Serletis (2000), and Barnett (2012) for more details regarding the Divisia approach to monetary aggregation. Over the years, the superior performance of the Divisia monetary aggregates has been demonstrated by a large number of studies, more recently by Barnett and Chauvet (2011), Serletis and
Gogas (2014), and Belongia and Ireland (2014, 2015), among others. However, the use of the Divisia index only partly solves the economic measurement problems associated with the failure in the literature to find significant relations between money and the economy, now widely referred to as the “Barnett critique.” In this regard, Jadidzadeh and Serletis (2019), use a very detailed demand system encompassing the full range of assets (also based on the normalized quadratic expenditure function) to estimate the degree of substitutability among assets and to address the issue of optimal monetary aggregation, as suggested by Barnett (1982, 2016). In doing so, they test for the appropriateness of the aggregation assumptions that underlie the various monetary aggregates published by the Federal Reserve, and reject the necessary and sufficient conditions for all the monetary aggregates published by the Federal Reserve as well as for other money measures suggested by other studies. They conclude that we have nothing to lose by using the broadest and most theoretically consistent monetary aggregate in the United States today, that being the Centre for Financial Stability Divisia M4 monetary aggregate. This is also corroborated by Dery and Serletis (2020) in their comprehensive comparison of narrow and broad Divisia monetary aggregates.

10 Conclusions

Our analysis of asset demands is based on a strong link between neoclassical microeconomic theory, aggregation theory, index number theory, and econometric implementation. The underlying economic theory of utility maximization emphasizes the joint nature of asset demand decisions while the econometric implementation of this interdependence involves the simultaneous estimation of systems of asset demand equations. In this paper, we provide a solution to the inter-related problems of estimation of asset demand functions, instability of money demand relations, and monetary aggregation. Instead of assuming that consumer preferences are fixed, we assume Markov regime switching, thus allowing for complicated nonlinear dynamics and sudden changes in the parameters of the underlying aggregator function. Moreover, using the normalized quadratic flexible functional form with Markov regime switching, we analytically prove the invariance of the maximum likelihood estimator with respect to the choice of the good deleted and the ordering of the remaining equations in the estimated subsystem.

We generate inference in terms of a full set of elasticities, treating the concavity property as a maintained hypothesis, and simultaneously achieving consistency with the data generating process, economic regularity, and econometric regularity. We find evidence that our five-regime normalized quadratic model provides a better fit of the actual data than a single regime model. We also find that the assets are in general Morishima substitutes, with the Morishima elasticities of substitution always being very low (in fact below one), and that the asset demand specifications exhibit instability (in terms of swings in the Morishima elasticities of substitution) during periods of rapid financial innovation, regulatory changes, and
erratic monetary policy, but they are rather stable within regimes. We also address the issue of monetary aggregation, and argue that the simple-sum monetary aggregates used by central banks are inadequate and misleading, because they view monetary assets as dollar-for-dollar perfect substitutes, an assumption we strongly reject.
11 Appendix 1

Proof of Theorem 1 Assume that the subsystem obtained by deleting the $i$th equation has (regime-dependent) error term $u_t$, and that obtained by deleting the $j$th equation has error term $e_t$, where $i, j \in \{1, \ldots, n\}$ and $i \neq j$. The subsystem that corresponds to $u_t$ is

$$s_t^u = s^u(\vartheta | v_t, z_t) + u_{t,z_t}$$
$$u_{t,z_t} \sim F (0, \Omega_{z_t}, g_{z_t} (u_{t,z_t}^{-1} u_{t,z_t})) .$$

Let $f(s_t^u | \Psi_{t-1})$ be the predictive density of $s_t^u$ in period $t$ given $\Psi_{t-1}$, which is the information set as of period $t - 1$. Moreover

$$f(s_t^u | \Psi_{t-1}) = \sum_{i=1}^{m} p^u(z_t = i | \Psi_{t-1}) f(s_t^u | z_t = i)$$

where $p^u(z_t = i | \Psi_{t-1})$ is the probability of regime $i$ at time $t$ given $\Psi_{t-1}$. According to Hamilton (1994), this probability is given by

$$p^u(z_t = i | \Psi_{t-1}) = \sum_{j=1}^{m} p_{ij} p^u(z_{t-1} = j | \Psi_{t-1}), \quad i = 1, \ldots, m.$$  \hspace{1cm} (21)

Moreover, it is known that

$$p^u(z_{t-1} = j | \Psi_{t-1}) = \frac{p^u(z_{t-1} = j | \Psi_{t-2}) f(s_{t-1}^u | z_{t-1} = j)}{\sum_{i=1}^{m} p^u(z_{t-1} = i | \Psi_{t-2}) f(s_{t-1}^u | z_{t-1} = i)}, \quad j = 1, \ldots, m.$$  \hspace{1cm} (22)

To start the above algorithm, we use the ergodic probability of the $m$-state Markov chain for $p^u(z_1 = j | \Psi_0)$, following Hamilton (1994).

The regime switching demand system can then be estimated by maximizing the following log likelihood function

$$\mathcal{L}^u (\vartheta) = \sum_{t=1}^{T} \log f(s_t^u | \Psi_{t-1})$$

where $T$ is the number of observations. Using (20) and (21), equation (23) and the log likelihood function (24) can be written as

$$p^u(z_{t-1} = j | \Psi_{t-1}) = \frac{p^u(z_{t-1} = j | \Psi_{t-2}) \left( | \Omega_{z_t}^{-1} \frac{1}{2} g_{z_t} (u'_{t-1,z_t-1} \Omega_{z_t-1}^{-1} u_{t-1,z_t-1}) | z_{t-1} = j \right)}{\sum_{i=1}^{m} p^u(z_{t-1} = i | \Psi_{t-2}) \left( | \Omega_{z_t}^{-1} \frac{1}{2} g_{z_t} (u'_{t-1,z_t-1} \Omega_{z_t-1}^{-1} u_{t-1,z_t-1}) | z_{t-1} = i \right)}$$

\hspace{1cm} (25)
where \( j = 1, \ldots, m \) and

\[
L^u(\theta) = \sum_{t=1}^{T} \log \left( \sum_{i=1}^{m} p^u(z_{t-1} = i \mid \Psi_{t-1}) \left( |\Omega_{z_t}|^{-\frac{1}{2}} g_{z_t} \left( u'_{t,z_t} \Omega_{z_t}^{-1} u_{t,z_t} \right) \mid z_t = i \right) \right). 
\]

(26)

Similarly, the subsystem that corresponds to \( e_t \) is

\[
s^e_t = s^e(\theta \mid \mathbf{v}_{t}, z_t) + e_{t,zt} 
\]

(27)

\[
e_{t,zt} \sim F(\mathbf{0}, \Phi_{z_t}, g_{z_t} \left( e'_{t,zt} \Phi_{z_t}^{-1} e_{t,zt} \right)) 
\]

(28)

with the log likelihood function

\[
L^e(\theta) = \sum_{t=1}^{T} \log \left( \sum_{i=1}^{m} p^e(z_t = i \mid \Psi_{t-1}) \left( |\Phi_{z_t}|^{-\frac{1}{2}} g_{z_t} \left( e'_{t,zt} \Phi_{z_t}^{-1} e_{t,zt} \right) \mid z_t = i \right) \right). 
\]

(29)

where

\[
p^e(z_t = i \mid \Psi_{t-1}) = \sum_{j=1}^{m} p_{ij}p^e(z_{t-1} = j \mid \Psi_{t-1}), \quad i = 1, \ldots, m
\]

(30)

and

\[
p^e(z_{t-1} = j \mid \Psi_{t-1}) = \frac{p^e(z_{t-1} = j \mid \Psi_{t-2}) \left( |\Phi_{z_{t-1}}|^{-\frac{1}{2}} g_{z_{t-1}} \left( e'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} e_{t-1,z_{t-1}} \right) \mid z_{t-1} = j \right)}{\sum_{i=1}^{m} p^e(z_{t-1} = i \mid \Psi_{t-2}) \left( |\Phi_{z_{t-1}}|^{-\frac{1}{2}} g_{z_{t-1}} \left( e'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} e_{t-1,z_{t-1}} \right) \mid z_{t-1} = i \right)}
\]

(31)

for \( j = 1, \ldots, m \).

Due to the adding-up property of the full demand system, there is a linear transformation between \( e_{t,zt} \) and \( u_{t,zt} \) as follows

\[
e_{t,zt} = Q u_{t,zt}
\]

(32)

where \( Q \) is a \((n - 1) \times (n - 1)\) matrix, which can be obtained easily. Let’s consider the \( n \times n \) identity matrix \( I_n \) and replace each of the elements of the \( j \)th row by \(-1\), and then delete the \( i \)th row and the \( i \)th column. The resulting matrix is \( Q \). It can be verified that \( Q \) has the following property

\[
QQ = I_{n-1}.
\]

(33)

According to (32) and (33), we have

\[
\Phi_{z_t} = Q \Omega_{z_t} \Phi'
\]

(34)
Applying equations (32)-(34) to equation (31), yields

$$p^e (z_{t-1} = j | \Psi_{t-2}) = \frac{p^e (z_{t-1} = j | \Psi_{t-2}) \left( | \Phi_{z_{t-1}} |^{-\frac{1}{2}} g_{z_{t-1}} \left( e'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} e_{t-1,z_{t-1}} \right) | z_{t-1} = j \right)}{\sum_{i=1}^{m} p^e (z_{t-1} = i | \Psi_{t-2}) \left( | \Phi_{z_{t-1}} |^{-\frac{1}{2}} g_{z_{t-1}} \left( e'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} e_{t-1,z_{t-1}} \right) | z_{t-1} = i \right)}$$

$$= \frac{p^e (z_{t-1} = j | \Psi_{t-2}) \left( | Q \Omega_{z_{t-1}} Q' |^{-\frac{1}{2}} g_{z_{t-1}} \left( (Q u_{t-1,z_{t-1}})' \left( Q \Omega_{z_{t-1}} Q' \right)^{-1} (Q u_{t-1,z_{t-1}}) \right) | z_{t-1} = j \right)}{\sum_{i=1}^{m} p^e (z_{t-1} = i | \Psi_{t-2}) \left( | Q \Omega_{z_{t-1}} Q' |^{-\frac{1}{2}} g_{z_{t-1}} \left( (Q u_{t-1,z_{t-1}})' \left( Q \Omega_{z_{t-1}} Q' \right)^{-1} (Q u_{t-1,z_{t-1}}) \right) | z_{t-1} = i \right)}$$

$$= \frac{p^e (z_{t-1} = j | \Psi_{t-2}) \left( \left( \Omega_{z_{t-1}} \right)^{-\frac{1}{2}} g_{z_{t-1}} \left( u'_{t-1,z_{t-1}} \Omega_{z_{t-1}}^{-1} u_{t-1,z_{t-1}} \right) | z_{t-1} = j \right)}{\sum_{i=1}^{m} p^e (z_{t-1} = i | \Psi_{t-2}) \left( \left( \Omega_{z_{t-1}} \right)^{-\frac{1}{2}} g_{z_{t-1}} \left( u'_{t-1,z_{t-1}} \Omega_{z_{t-1}}^{-1} u_{t-1,z_{t-1}} \right) | z_{t-1} = i \right)} \quad (35)$$

Since the same ergodic probability of the m-state Markov chain is also used for \( p^e (z_1 = j | \Psi_0) \), equation (35) suggests

$$p^e (z_1 = j | \Psi_0) = p^u (z_1 = j | \Psi_0), \quad j = 1, ..., m \quad (36)$$

and it follows that

$$p^e (z_t = j | \Psi_{t-1}) = p^u (z_t = j | \Psi_{t-1}), \quad j = 1, ..., m \quad (37)$$

for all time periods.

Using equations (32)-(34) and (37), the log likelihood function (29) can be written as

$$L^e (\vartheta) = \sum_{t=1}^{T} \log \left( \sum_{i=1}^{m} p^e (z_t = i | \Psi_{t-1}) \left( | Q \Omega_{z_{t-1}} Q' |^{-\frac{1}{2}} g_{z_{t}} \left( (Q u_{t,z_{t}})' \left( Q \Omega_{z_{t}} Q' \right) (Q u_{t,z_{t}}) \right) | z_{t} = i \right) \right)$$

$$= \sum_{t=1}^{T} \log \left( \sum_{i=1}^{m} p^u (z_t = i | \Psi_{t-1}) \left( \left( \Omega_{z_{t}} \right)^{-\frac{1}{2}} g_{z_{t}} \left( u'_{t,z_{t}} \Omega_{z_{t}}^{-1} u_{t,z_{t}} \right) | z_{t} = i \right) \right)$$

$$= L^u (\vartheta)$$

according to which different demand subsystems, obtained by deleting different equations (or, equivalently, goods), have the same log likelihood function. Q.E.D.
12 Appendix 2

Proof of Theorem 2 Consider a Markov switching demand system and let $e_{t,z}$ denote the error term of the subsystem obtained by deleting the $j$th equation. Let $\tilde{e}_{t,z}$ denote the error term of another subsystem obtained by deleting again the $j$th equation but with the ordering of the remaining equations being different than that in $e_{t,z}$. We introduce a $(n-1) \times (n-1)$ sorting matrix $S$ so that

$$\tilde{e}_{t,z} = S e_{t,z}.$$  

(38)

The sorting matrix $S$ could be found by modifying a $(n-1) \times (n-1)$ null matrix $0$. If the $l$th element in $e_{t,z}$ is the $m$th element in $e_{t,z}$, we then change the element in the $m$th row and $l$th column of the null matrix $0$ to 1, where $m,l \in (1, \ldots, n-1)$. It can be shown that $S$ is equivalent to a $(n-1) \times (n-1)$ identity matrix after some elementary transformations. Thus, $S$ is full rank and is invertible and it is a simple matter to verify that

$$SS' = I_{n-1}$$

and

$$|S| \in \{-1, 1\}.$$

According to equation (38), we have

$$\tilde{\Phi}_{t,z} = S \Phi_{t,z} S'$$

(39)

where $\tilde{\Phi}_{t,z}$ is the covariance matrix of $\tilde{e}_{t,z}$. The log likelihood function corresponding to $\tilde{\Phi}_{t,z}$ includes

$$L^\tilde{e}(\theta) = \sum_{t=1}^{T} \log \left( \sum_{i=1}^{m} p^\tilde{e}(z_t = i | \Psi_{t-1}) \left( \tilde{\Phi}_{t,z} \left| g_{zt} \left( \tilde{e}'_{t,z} \tilde{\Phi}_{t,z} \tilde{e}_{t,z} \right) | z_t = i \right| \right) \right)$$

(40)

where

$$p^\tilde{e}(z_t = i | \Psi_{t-1}) = \sum_{j=1}^{m} p_{ij} p_j^\tilde{e}(z_{t-1} = j | \Psi_{t-1}), \quad i = 1, \ldots, m$$

(41)

and

$$p^\tilde{e}(z_{t-1} = j | \Psi_{t-1}) = \frac{p^\tilde{e}(z_{t-1} = j | \Psi_{t-2}) \left( | \tilde{\Phi}_{zt-1} \left| g_{zt-1} \left( \tilde{e}'_{t-1,zt-1} \tilde{\Phi}_{zt-1} \tilde{e}_{t-1,zt-1} \right) | z_{t-1} = j \right| \right)}{\sum_{i=1}^{m} p^\tilde{e}(z_{t-1} = i | \Psi_{t-2}) \left( | \tilde{\Phi}_{zt-1} \left| g_{zt-1} \left( \tilde{e}'_{t-1,zt-1} \tilde{\Phi}_{zt-1} \tilde{e}_{t-1,zt-1} \right) | z_{t-1} = i \right| \right)}$$

(42)

where $j = 1, \ldots, m$. 

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Using (38) and (39) in (42) yields

\[
p^\tilde{e}(z_{t-1} = j \mid \Psi_{t-1}) = \frac{\prod_{i=1}^m p^\tilde{e}(z_{t-1} = i \mid \Psi_{t-2}) \left[ |\Phi_{z_{t-1}}|^{-\frac{1}{2}} g_{z_{t-1}} \left( \tilde{e}'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} \tilde{e}_{t-1,z_{t-1}} \right) |z_{t-1} = j \right]}{\sum_{i=1}^m p^\tilde{e}(z_{t-1} = i \mid \Psi_{t-2}) \left[ |\Phi_{z_{t-1}}|^{-\frac{1}{2}} g_{z_{t-1}} \left( \tilde{e}'_{t-1,z_{t-1}} \Phi_{z_{t-1}}^{-1} \tilde{e}_{t-1,z_{t-1}} \right) |z_{t-1} = i \right]}
\]

\[= \frac{\prod_{i=1}^m p^\tilde{e}(z_{t-1} = j \mid \Psi_{t-2}) \left[ |S\Phi_{z_{t-1}}S'|^{-\frac{1}{2}} g_{z_{t-1}} \left( (Se_{t-1,z_{t-1}})'(S\Phi_{z_{t-1}}S')^{-1}(Se_{t-1,z_{t-1}}) \right) |z_{t-1} = j \right]}{\sum_{i=1}^m p^\tilde{e}(z_{t-1} = i \mid \Psi_{t-2}) \left[ |S\Phi_{z_{t-1}}S'|^{-\frac{1}{2}} g_{z_{t-1}} \left( (Se_{t-1,z_{t-1}})'(S\Phi_{z_{t-1}}S')^{-1}(Se_{t-1,z_{t-1}}) \right) |z_{t-1} = i \right]}.
\]

(43)

Since the ergodic probability of the m-state Markov chain is used for \(p^e(z_1 = j \mid \Psi_0)\) and \(p^\tilde{e}(z_1 = j \mid \Psi_0)\), equation (43) suggests that

\[
p^e(z_1 = j \mid \Psi_0) = p^\tilde{e}(z_1 = j \mid \Psi_0), \quad j = 1, \ldots, m
\]

and it follows that

\[
p^e(z_t = j \mid \Psi_{t-1}) = p^\tilde{e}(z_t = j \mid \Psi_{t-1}), \quad j = 1, \ldots, m
\]

(45)

for all time periods. Now using (38), (39) and (45) in (40) yields

\[
\mathcal{L}^\tilde{e}(\theta) = \sum_{t=1}^T \log \left( \sum_{i=1}^m p^\tilde{e}(z_t = i \mid \Psi_{t-1}) \left[ |S\Phi_{t,z_t}S'|^{-\frac{1}{2}} g_{z_t} \left( (Se_{t,z_t})'(S\Phi_{t,z_t}S')^{-1}(Se_{t,z_t}) \right) |z_t = i \right] \right)
\]

\[= \sum_{t=1}^T \log \left( \sum_{i=1}^m p^e(z_t = i \mid \Psi_{t-1}) \left[ \Phi_{t,z_t}^{-\frac{1}{2}} g_{z_t} \left( e'_{t,z_t}\Phi_{t,z_t}^{-1}e_{t,z_t} \right) |z_t = i \right] \right)
\]

\[= \mathcal{L}^e(\theta)
\]

which demonstrates invariance under the maximum likelihood estimator. Q.E.D.
References


Table 1. Monetary assets/components

<table>
<thead>
<tr>
<th>Mnemonic</th>
<th>Asset/component</th>
<th>Monetary subaggregate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>Currency</td>
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</tr>
<tr>
<td>$x_2$</td>
<td>Traveler’s check</td>
<td>✓</td>
</tr>
<tr>
<td>$x_3$</td>
<td>Demand deposits</td>
<td>✓</td>
</tr>
<tr>
<td>$x_4$</td>
<td>OCDs at commercial banks</td>
<td>✓</td>
</tr>
<tr>
<td>$x_5$</td>
<td>OCDs at thrift institutions</td>
<td>✓</td>
</tr>
<tr>
<td>$x_6$</td>
<td>Saving deposits at banks including MMDAs</td>
<td>✓</td>
</tr>
<tr>
<td>$x_7$</td>
<td>Saving deposits at thrifts including MMDAs</td>
<td>✓</td>
</tr>
<tr>
<td>$x_8$</td>
<td>Retail money-market funds</td>
<td>✓</td>
</tr>
<tr>
<td>$x_9$</td>
<td>Small time deposits at commercial banks</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>Small time deposits at thrift institutions</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>Institutional money-market funds</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{12}$</td>
<td>Large time deposits</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>Repurchase agreements</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{14}$</td>
<td>Commercial paper</td>
<td>✓</td>
</tr>
<tr>
<td>$x_{15}$</td>
<td>T-bills</td>
<td>✓</td>
</tr>
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<td>Normalized quadratic model</td>
<td>BIC</td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------------------------</td>
<td>-----------</td>
<td></td>
</tr>
<tr>
<td>Single regime</td>
<td>-7810.7740</td>
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</tr>
<tr>
<td>Two regimes</td>
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<td></td>
</tr>
<tr>
<td>Two regimes with theoretical regularity imposed</td>
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</tr>
<tr>
<td>Three regimes</td>
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<td></td>
</tr>
<tr>
<td>Three regimes with theoretical regularity imposed</td>
<td>-10857.2905</td>
<td></td>
</tr>
<tr>
<td>Four regimes</td>
<td>-11842.5838</td>
<td></td>
</tr>
<tr>
<td>Four regimes with theoretical regularity imposed</td>
<td>-11430.5812</td>
<td></td>
</tr>
<tr>
<td>Five regimes</td>
<td>-12001.5397</td>
<td></td>
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<tr>
<td>Five regimes with theoretical regularity imposed</td>
<td>-11942.2785</td>
<td></td>
</tr>
<tr>
<td>Six regimes</td>
<td>-12067.0457</td>
<td></td>
</tr>
<tr>
<td>Six regimes with theoretical regularity imposed</td>
<td>-11653.4407</td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. Smoothed probabilities of five regimes
Table 3. Parameter estimates for the Normalized Quadratic functions with curvature imposed

Assets:
1 = q_1 (Divisia M1)
2 = q_2 (Saving and small time deposits)
3 = q_3 (Institutional MMF, large time deposits, and repurchase agreements)
4 = q_4 (Commercial paper and T-bills)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Single regime NQ</th>
<th>Regime 1</th>
<th>Regime 2</th>
<th>Regime 3</th>
<th>Regime 4</th>
<th>Regime 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 )</td>
<td>0.1256 (0.0110)</td>
<td>0.0796 (0.0049)</td>
<td>0.1723 (0.0033)</td>
<td>0.2000 (0.0145)</td>
<td>0.0945 (0.0191)</td>
<td>-0.0289 (0.0048)</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>0.1057 (0.0054)</td>
<td>0.1472 (0.0051)</td>
<td>0.0706 (0.0038)</td>
<td>-0.1942 (0.0243)</td>
<td>0.3349 (0.0232)</td>
<td>0.0170 (0.0051)</td>
</tr>
<tr>
<td>( \theta_3 )</td>
<td>-0.2886 (0.0075)</td>
<td>-0.2440 (0.0131)</td>
<td>-0.4462 (0.0107)</td>
<td>-0.0621 (0.0388)</td>
<td>-0.3699 (0.0183)</td>
<td>-0.0556 (0.0063)</td>
</tr>
<tr>
<td>( \theta_4 )</td>
<td>0.0573 (0.0068)</td>
<td>0.0173 (0.0091)</td>
<td>0.2033 (0.0114)</td>
<td>0.0563 (0.0168)</td>
<td>-0.0595 (0.0270)</td>
<td>0.0674 (0.0040)</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>0.0679 (0.0075)</td>
<td>0.0999 (0.0055)</td>
<td>0.0322 (0.0026)</td>
<td>-0.0134 (0.0186)</td>
<td>0.1129 (0.0207)</td>
<td>0.2564 (0.0056)</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.1166 (0.0051)</td>
<td>0.0848 (0.0056)</td>
<td>0.1204 (0.0029)</td>
<td>0.5788 (0.0311)</td>
<td>-0.1509 (0.0253)</td>
<td>0.2177 (0.0054)</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0.6622 (0.0075)</td>
<td>0.6292 (0.0140)</td>
<td>0.7979 (0.0077)</td>
<td>0.3584 (0.0496)</td>
<td>0.7030 (0.0200)</td>
<td>0.4187 (0.0068)</td>
</tr>
<tr>
<td>( b_4 )</td>
<td>0.1533 (0.0066)</td>
<td>0.1861 (0.0104)</td>
<td>0.0495 (0.0082)</td>
<td>0.0763 (0.0213)</td>
<td>0.3350 (0.0293)</td>
<td>0.1072 (0.0047)</td>
</tr>
<tr>
<td>( \beta_{11} )</td>
<td>-0.2181 (0.0508)</td>
<td>-0.1110 (0.0121)</td>
<td>-0.0283 (0.0079)</td>
<td>-0.1263 (0.0192)</td>
<td>-0.1736 (0.0089)</td>
<td>-0.0542 (0.0024)</td>
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<tr>
<td>( \beta_{12} )</td>
<td>0.1638 (0.0129)</td>
<td>0.0208 (0.0071)</td>
<td>0.0093 (0.0078)</td>
<td>0.0288 (0.0159)</td>
<td>0.2406 (0.0077)</td>
<td>0.0569 (0.0020)</td>
</tr>
<tr>
<td>( \beta_{13} )</td>
<td>0.0133 (0.0128)</td>
<td>0.0686 (0.0098)</td>
<td>0.0612 (0.0051)</td>
<td>0.0518 (0.0157)</td>
<td>0.0390 (0.0102)</td>
<td>-0.0105 (0.0083)</td>
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<td>( \beta_{14} )</td>
<td>0.0409 (0.0141)</td>
<td>0.0216 (0.0125)</td>
<td>-0.0422 (0.0043)</td>
<td>0.0456 (0.0154)</td>
<td>-0.1059 (0.0160)</td>
<td>0.0078 (0.0101)</td>
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<tr>
<td>( \beta_{22} )</td>
<td>-0.2915 (0.0576)</td>
<td>-0.1132 (0.0089)</td>
<td>-0.0446 (0.0088)</td>
<td>-0.3224 (0.0252)</td>
<td>-0.3425 (0.0184)</td>
<td>-0.0784 (0.0185)</td>
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<tr>
<td>( \beta_{23} )</td>
<td>0.0612 (0.0014)</td>
<td>0.0516 (0.0095)</td>
<td>0.0476 (0.0086)</td>
<td>0.3525 (0.0258)</td>
<td>-0.0587 (0.0127)</td>
<td>-0.0290 (0.0093)</td>
</tr>
<tr>
<td>( \beta_{24} )</td>
<td>0.0665 (0.0131)</td>
<td>0.0408 (0.0116)</td>
<td>-0.0124 (0.0078)</td>
<td>-0.0590 (0.0140)</td>
<td>0.1607 (0.0202)</td>
<td>0.0505 (0.0111)</td>
</tr>
<tr>
<td>( \beta_{33} )</td>
<td>-0.2896 (0.0396)</td>
<td>-0.0804 (0.0205)</td>
<td>-0.2428 (0.0404)</td>
<td>-0.4417 (0.0635)</td>
<td>-0.0112 (0.0133)</td>
<td>-0.0878 (0.0362)</td>
</tr>
<tr>
<td>( \beta_{34} )</td>
<td>0.2151 (0.0206)</td>
<td>-0.0398 (0.0179)</td>
<td>0.1340 (0.0375)</td>
<td>0.0373 (0.0525)</td>
<td>0.0310 (0.0196)</td>
<td>0.1273 (0.0329)</td>
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<tr>
<td>( \beta_{44} )</td>
<td>-0.3225 (0.0302)</td>
<td>-0.0226 (0.0142)</td>
<td>-0.0794 (0.0368)</td>
<td>-0.0240 (0.0520)</td>
<td>-0.0858 (0.0359)</td>
<td>-0.1857 (0.0356)</td>
</tr>
</tbody>
</table>

![Transition matrix](https://example.com/transition_matrix)

\[
\begin{bmatrix}
0.9815 (0.0116) & 0.0067 (0.0053) & 0.0000 (0.0000) & 0.0000 (0.0000) & 0.0058 (0.0058) \\
0.0000 (0.0000) & 0.9933 (0.0053) & 0.0000 (0.0000) & 0.0126 (0.0118) & 0.0000 (0.0000) \\
0.0000 (0.0000) & 0.0000 (0.0000) & 0.9947 (0.0064) & 0.0000 (0.0000) & 0.0061 (0.0068) \\
0.0067 (0.0072) & 0.0000 (0.0000) & 0.0000 (0.0000) & 0.9874 (0.0118) & 0.0000 (0.0000) \\
0.0118 (0.0084) & 0.0000 (0.0000) & 0.0053 (0.0064) & 0.0000 (0.0000) & 0.9881 (0.0093)
\end{bmatrix}
\]

*Note: Numbers in parentheses are standard errors.*
Table 4. Morishima elasticities of substitution

<table>
<thead>
<tr>
<th>Regime</th>
<th>Monetary subaggregate $i$</th>
<th>Morishima elasticities of substitution</th>
<th>$\sigma_{1i}^n$</th>
<th>$\sigma_{2i}^n$</th>
<th>$\sigma_{3i}^n$</th>
<th>$\sigma_{4i}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A. Single regime NQ model</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td></td>
<td>0.4634 (0.0000)</td>
<td>0.1846 (0.0000)</td>
<td>0.4947 (0.0000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.4019 (0.0000)</td>
<td>0.2336 (0.0000)</td>
<td>0.5112 (0.0000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td>0.2586 (0.0000)</td>
<td>0.3247 (0.0000)</td>
<td></td>
<td>0.5981 (0.0000)</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td>0.3128 (0.0000)</td>
<td>0.3773 (0.0000)</td>
<td>0.5215 (0.0000)</td>
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</tr>
<tr>
<td></td>
<td>B. Markov switching NQ model</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One</td>
<td>(1)</td>
<td></td>
<td>0.1137 (0.0000)</td>
<td>0.1552 (0.0000)</td>
<td>0.0605 (0.0903)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.1558 (0.0000)</td>
<td>0.1049 (0.0000)</td>
<td>0.0671 (0.0330)</td>
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<tr>
<td></td>
<td>(3)</td>
<td></td>
<td>0.1908 (0.0000)</td>
<td>0.1275 (0.0000)</td>
<td>-0.0011 (0.9650)</td>
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</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td>0.1744 (0.0000)</td>
<td>0.1375 (0.0000)</td>
<td>0.0126 (0.6935)</td>
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<tr>
<td>Two</td>
<td>(1)</td>
<td></td>
<td>0.0800 (0.0006)</td>
<td>0.2820 (0.0000)</td>
<td>0.0380 (0.4655)</td>
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</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.0507 (0.0110)</td>
<td>0.2400 (0.0000)</td>
<td>0.3689 (0.0000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td>0.0693 (0.0000)</td>
<td>0.0932 (0.0000)</td>
<td>0.2077 (0.0051)</td>
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</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td>-0.0072 (0.5727)</td>
<td>0.0473 (0.0005)</td>
<td>0.3689 (0.0000)</td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>(1)</td>
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<td>0.1670 (0.0000)</td>
<td>0.4302 (0.0000)</td>
<td>0.1218 (0.2308)</td>
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</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.1385 (0.0000)</td>
<td>0.6053 (0.0000)</td>
<td>-0.0082 (0.9394)</td>
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<tr>
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<td>(3)</td>
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<td>0.1975 (0.0000)</td>
<td>0.3894 (0.0000)</td>
<td>0.1395 (0.4051)</td>
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</tr>
<tr>
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<td>(4)</td>
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<td>0.1986 (0.0000)</td>
<td>0.0814 (0.0159)</td>
<td>0.4514 (0.0060)</td>
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<tr>
<td>Four</td>
<td>(1)</td>
<td></td>
<td>0.6072 (0.0000)</td>
<td>0.0675 (0.0003)</td>
<td>-0.0469 (0.3311)</td>
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<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.4341 (0.0000)</td>
<td>-0.0778 (0.0000)</td>
<td>0.2682 (0.0000)</td>
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</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td>0.2187 (0.0000)</td>
<td>0.3028 (0.0000)</td>
<td>0.1105 (0.0400)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td>0.1006 (0.0000)</td>
<td>0.4828 (0.0000)</td>
<td>0.0450 (0.2014)</td>
<td></td>
</tr>
<tr>
<td>Five</td>
<td>(1)</td>
<td></td>
<td>0.1133 (0.0002)</td>
<td>0.0444 (0.0731)</td>
<td>0.2769 (0.0000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td></td>
<td>0.0723 (0.0189)</td>
<td>0.0173 (0.5119)</td>
<td>0.3195 (0.0000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td></td>
<td>0.0270 (0.1050)</td>
<td>0.0549 (0.0056)</td>
<td>0.3540 (0.0000)</td>
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</tr>
<tr>
<td></td>
<td>(4)</td>
<td></td>
<td>0.0577 (0.0108)</td>
<td>0.1407 (0.0000)</td>
<td>0.2736 (0.0001)</td>
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</tr>
</tbody>
</table>

*Note:* Numbers in parentheses are $p$ values.
Figure 2. Morishima elasticities of substitution for $q_1$ (Divisia M1)
Figure 3. Morishima elasticities of substitution for $q_2$
Figure 4. Morishima elasticities of substitution for $q_3$
Figure 5. Morishima elasticities of substitution for $q_4$
Table 5. Model comparisons (based on the BIC) with the NewM1

<table>
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<tr>
<th>Normalized quadratic model</th>
<th>BIC</th>
</tr>
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<tr>
<td>Single regime</td>
<td>−7013.8543</td>
</tr>
<tr>
<td>Two regimes</td>
<td>−9009.4956</td>
</tr>
<tr>
<td>Two regimes with theoretical regularity imposed</td>
<td>−8972.5506</td>
</tr>
<tr>
<td>Three regimes</td>
<td>−10745.6510</td>
</tr>
<tr>
<td>Three regimes with theoretical regularity imposed</td>
<td>−10540.1781</td>
</tr>
<tr>
<td>Four regimes</td>
<td>−11666.2659</td>
</tr>
<tr>
<td>Four regimes with theoretical regularity imposed</td>
<td>−11506.4938</td>
</tr>
<tr>
<td>Five regimes</td>
<td>−11459.5766</td>
</tr>
<tr>
<td>Five regimes with theoretical regularity imposed</td>
<td>−11429.7543</td>
</tr>
</tbody>
</table>
Figure 6. Smoothed probabilities of the four-regimes NQ model with the Divisia NewM1 aggregate
Figure 7. Morishima elasticities of substitution for the Divisia NewM1 aggregate
Figure 8. Morishima elasticities of substitution for $q_2$ with the Divisia NewM1 aggregate.
Figure 9. Morishima elasticities of substitution for $q_3$ with the Divisia NewM1 aggregate
Figure 10. Morishima elasticities of substitution for $q_4$ with the Divisia NewM1 aggregate
The Demand for Assets: Evidence from the Markov Switching Normalized Quadratic Model

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Canada

and

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Department of Economics
University of Calgary
Calgary, Alberta T2N 1N4
Canada

Online Appendix
August 21, 2021

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1 Elasticities

In this on-line Appendix, we present the income elasticities, the own- and cross-price elasticities, as well as the Allen-Uzawa elasticities of substitution.

1.1 Own- and cross-price elasticities

We present the income elasticities and the own- and cross-price elasticities in Table A1 (along with p-values), evaluated at the mean of the data, for each of the four monetary assets, $q_1$ (Divisia M1), $q_2$, $q_3$, and $q_4$, and (for comparison purposes) the two models, the single regime NQ (in panel A) and the Markov switching NQ (in panel B). The income elasticities ($\eta_{iy}$) are mostly positive and significant, suggesting that the monetary assets are normal goods in general. Although there are differences between the two models, and also across regimes in the case of the Markov switching NQ model, all income elasticities are in general less than 1 (implying economies of scale in liquid asset holdings), with the exception of $\eta_3$ which is typically greater than 1 (implying that $q_3$ is a luxury good).

Regarding the price elasticities, all own-price elasticities ($\eta_{ii}$), evaluated at the mean of the data, are negative, as predicted by the theory, and the cross-price elasticities ($\eta_{ij}$) indicate that in general the monetary assets are (gross) complements and that the relationship between the monetary assets is asymmetric. In Figure A1 we plot the complete set of the income elasticities and in Figures A2-A5 the own- and cross-price elasticities. The red lines indicate the elasticities based on the Markov switching NQ model and the black lines the corresponding elasticities based on the single regime NQ model. Shaded areas indicate the five regimes.

1.2 Allen-Uzawa elasticities

The Allen-Uzawa elasticities of substitution are reported in Table A2 (with p-values in parentheses), in the same fashion as for the own-and cross-price elasticities in Table A1. We also plot the complete set of the Allen-Uzawa elasticities in Figures A6-A9. In Table A2, the diagonal terms, representing the Allen-Uzawa own-elasticities of substitution for the four assets, are all negative (except for one which is not statistically significant), consistent with the theory.
Table A1. Marshallian income elasticities and own- and cross-price elasticities

<table>
<thead>
<tr>
<th>Regime subaggregate i</th>
<th>Monetary Income elasticity $\eta_{iy}$</th>
<th>Income elasticity $\eta_{iy}$</th>
<th>Own- and cross-price elasticities $\eta_1$</th>
<th>$\eta_2$</th>
<th>$\eta_3$</th>
<th>$\eta_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Single regime NQ model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) 0.3532 (0.0000)</td>
<td>−0.3119 (0.0000)</td>
<td>0.0999 (0.0001)</td>
<td>−0.1425 (0.0001)</td>
<td>0.0013 (0.9651)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2) 0.5518 (0.0000)</td>
<td>0.0617 (0.0347)</td>
<td>−0.4006 (0.0000)</td>
<td>−0.1826 (0.0000)</td>
<td>−0.0302 (0.1891)</td>
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<td></td>
</tr>
<tr>
<td>(3) 1.5153 (0.0000)</td>
<td>−0.2343 (0.0000)</td>
<td>−0.2554 (0.0000)</td>
<td>−0.8491 (0.0000)</td>
<td>−0.1765 (0.0000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) 0.6486 (0.0000)</td>
<td>−0.0470 (0.0178)</td>
<td>−0.0487 (0.0093)</td>
<td>0.0119 (0.7646)</td>
<td>−0.5648 (0.0000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B. Markov switching NQ model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One (1) 0.5307 (0.0000)</td>
<td>−0.2397 (0.0003)</td>
<td>−0.0854 (0.0000)</td>
<td>−0.1149 (0.0000)</td>
<td>−0.0907 (0.0001)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2) 0.3820 (0.0000)</td>
<td>−0.0570 (0.0000)</td>
<td>−0.1727 (0.0000)</td>
<td>−0.1043 (0.0000)</td>
<td>−0.0480 (0.0027)</td>
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</tr>
<tr>
<td>(3) 1.5321 (0.0000)</td>
<td>−0.2129 (0.0000)</td>
<td>−0.2445 (0.0000)</td>
<td>−0.6801 (0.0000)</td>
<td>−0.3946 (0.0000)</td>
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<td></td>
</tr>
<tr>
<td>(4) 0.8773 (0.0000)</td>
<td>−0.1200 (0.0000)</td>
<td>−0.1206 (0.0000)</td>
<td>−0.4017 (0.0000)</td>
<td>−0.2350 (0.0000)</td>
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<td></td>
</tr>
<tr>
<td>Two (1) 0.2218 (0.0000)</td>
<td>−0.0682 (0.0000)</td>
<td>−0.0248 (0.0950)</td>
<td>0.0050 (0.7145)</td>
<td>−0.1338 (0.0000)</td>
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<tr>
<td>(2) 0.6974 (0.0000)</td>
<td>−0.0704 (0.0000)</td>
<td>−0.1805 (0.0000)</td>
<td>−0.4295 (0.0000)</td>
<td>−0.0279 (0.0000)</td>
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<tr>
<td>(3) 1.5864 (0.0000)</td>
<td>−0.1504 (0.0000)</td>
<td>−0.2763 (0.0000)</td>
<td>−0.9736 (0.0000)</td>
<td>−0.2361 (0.0000)</td>
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</tr>
<tr>
<td>(4) 0.2726 (0.0000)</td>
<td>−0.0811 (0.0000)</td>
<td>−0.0664 (0.0000)</td>
<td>0.0567 (0.3757)</td>
<td>−0.1819 (0.0004)</td>
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</tr>
<tr>
<td>Three (1) −0.0546 (0.5270)</td>
<td>−0.1164 (0.0003)</td>
<td>0.0207 (0.4160)</td>
<td>0.0798 (0.0017)</td>
<td>0.0705 (0.0079)</td>
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</tr>
<tr>
<td>(2) 1.9496 (0.0000)</td>
<td>−0.5126 (0.0000)</td>
<td>−0.6476 (0.0000)</td>
<td>−0.2708 (0.0000)</td>
<td>0.5187 (0.0000)</td>
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</tr>
<tr>
<td>(3) 1.4968 (0.0000)</td>
<td>−0.3329 (0.0000)</td>
<td>−0.1394 (0.0252)</td>
<td>−0.7573 (0.0000)</td>
<td>−0.2672 (0.0028)</td>
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<td></td>
</tr>
<tr>
<td>(4) 0.5865 (0.0003)</td>
<td>−0.0888 (0.1317)</td>
<td>−0.2283 (0.0000)</td>
<td>−0.0713 (0.4332)</td>
<td>−0.1981 (0.0197)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Four (1) 0.4713 (0.0000)</td>
<td>−0.2751 (0.0000)</td>
<td>0.1576 (0.0000)</td>
<td>−0.0894 (0.0223)</td>
<td>−0.2644 (0.0000)</td>
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</tr>
<tr>
<td>(2) −0.5613 (0.0000)</td>
<td>0.3621 (0.0000)</td>
<td>−0.2478 (0.0000)</td>
<td>0.0878 (0.0193)</td>
<td>0.3592 (0.0000)</td>
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<tr>
<td>(3) 2.1428 (0.0000)</td>
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<td>−0.4705 (0.0000)</td>
<td>−0.6791 (0.0000)</td>
<td>−0.6065 (0.0000)</td>
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<tr>
<td>(4) 1.1626 (0.0000)</td>
<td>−0.3110 (0.0000)</td>
<td>−0.1004 (0.0002)</td>
<td>−0.3270 (0.0000)</td>
<td>−0.4241 (0.0000)</td>
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<tr>
<td>Five (1) 0.3579 (0.0000)</td>
<td>−0.0574 (0.0017)</td>
<td>0.0751 (0.0000)</td>
<td>−0.2438 (0.0000)</td>
<td>−0.1318 (0.0000)</td>
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<tr>
<td>(2) −0.5033 (0.0000)</td>
<td>0.0686 (0.0002)</td>
<td>−0.1095 (0.0000)</td>
<td>0.2954 (0.0000)</td>
<td>0.2487 (0.0000)</td>
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<tr>
<td>(3) 1.4452 (0.0000)</td>
<td>−0.0921 (0.0000)</td>
<td>0.1325 (0.0000)</td>
<td>−0.9903 (0.0000)</td>
<td>−0.4953 (0.0000)</td>
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<tr>
<td>(4) 1.4129 (0.0000)</td>
<td>−0.0755 (0.0026)</td>
<td>0.1934 (0.0000)</td>
<td>−0.7578 (0.0000)</td>
<td>−0.7730 (0.0000)</td>
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</table>

*Note*: Numbers in parentheses are $p$ values.
Figure A1. Income elasticities
Figure A2. Marshallian own- and cross-price elasticities for $q_1$ (Divisia M1)
Figure A3. Marshallian own- and cross-price elasticities for $q_2$
Figure A4. Marshallian own- and cross-price elasticities for $q_3$
Figure A5. Marshallian own- and cross-price elasticities for $q_4$. 

[Graphs showing Marshallian own- and cross-price elasticities for $q_4$ with different regimes indicated.]
<table>
<thead>
<tr>
<th>Regime</th>
<th>Monetary subaggregate $i$</th>
<th>Allen-Uzawa elasticities of substitution</th>
<th>$\sigma_{i1}^a$</th>
<th>$\sigma_{i2}^a$</th>
<th>$\sigma_{i3}^a$</th>
<th>$\sigma_{i4}^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Single regime NQ model</strong></td>
<td></td>
<td></td>
<td>(-1.5025 (0.0000))</td>
<td>(0.8883 (0.0000))</td>
<td>(0.0361 (0.4608))</td>
<td>(0.3586 (0.0004))</td>
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<td></td>
<td>(-1.5945 (0.0000))</td>
<td>(-0.3749 (0.0000))</td>
<td>(-0.0361 (0.4608))</td>
<td>(-1.6854 (0.0000))</td>
</tr>
<tr>
<td><strong>B. Markov switching NQ model</strong></td>
<td></td>
<td></td>
<td>(-0.8436 (0.0000))</td>
<td>(0.0496 (0.3106))</td>
<td>(0.2500 (0.0000))</td>
<td>(0.1561 (0.0000))</td>
</tr>
<tr>
<td>One</td>
<td></td>
<td></td>
<td>(-0.5911 (0.0000))</td>
<td>(0.1272 (0.0000))</td>
<td>(-0.0983 (0.0193))</td>
<td>(-0.0938 (0.2697))</td>
</tr>
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<td></td>
<td>(-0.3854 (0.0001))</td>
<td>(0.0662 (0.4310))</td>
<td>(0.2315 (0.0000))</td>
<td>(-0.4493 (0.0000))</td>
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<tr>
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<td>(-0.3211 (0.0000))</td>
<td>(0.4018 (0.0003))</td>
<td>(-0.6400 (0.0148))</td>
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<td>(-0.4907 (0.0000))</td>
<td>(0.0282 (0.5452))</td>
<td>(0.2495 (0.0003))</td>
<td>(0.2532 (0.0050))</td>
</tr>
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<td>Three</td>
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<td>(-0.6397 (0.0000))</td>
<td>(-1.3905 (0.0000))</td>
<td>(0.3305 (0.3464))</td>
<td>(-0.2782 (0.4730))</td>
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<td>(-0.9238 (0.0000))</td>
<td>(1.2775 (0.0000))</td>
<td>(0.1851 (0.0000))</td>
<td>(-0.4135 (0.0000))</td>
</tr>
<tr>
<td>Four</td>
<td></td>
<td></td>
<td>(-1.8289 (0.0000))</td>
<td>(-0.310 (0.4920))</td>
<td>(0.1131 (0.1117))</td>
<td>(-0.2567 (0.0234))</td>
</tr>
<tr>
<td></td>
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<td>(-0.219 (0.7641))</td>
<td>(0.2961 (0.0000))</td>
<td>(0.0966 (0.0002))</td>
<td>(0.2320 (0.0000))</td>
</tr>
<tr>
<td>Five</td>
<td></td>
<td></td>
<td>(-0.1981 (0.0061))</td>
<td>(0.0413 (0.1061))</td>
<td>(0.4161 (0.0000))</td>
<td>(0.5652 (0.0000))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0059 (0.8973))</td>
<td>(-0.9642 (0.0000))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: Numbers in parentheses are $p$ values.*
Figure A6. Allen-Uzawa elasticities of substitution for $q_1$ (Divisia M1)
Figure A7. Allen-Uzawa elasticities of substitution for $q_2$
Figure A8. Allen-Uzawa elasticities of substitution for $q_3$. 
Figure A9. Allen-Uzawa elasticities of substitution for $q_4$